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Positive implicative vague ideals in BCK-algebras

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Abstract. The notion of positive implicative vague ideals in BCK-algebras is introduced. A relation between a vague ideal and a positive implicative vague ideal is discussed. Characterizations of a positive implicative vague ideal are considered.

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1. Introduction

Several authors have made a number of generalizations of Zadeh's fuzzy set theory [10]. Of these, the notion of vague set theory introduced by Gau and Buehrer [3] is of interest to us. Using the vague set in the sense of Gau and Buehrer, Biswas [2] studied vague groups. Jun and Park [5, 9] studied vague ideals and vague deductive systems in subtraction algebras. Lee et al. [7] introduced the notion of vague BCK/BCI-algebras and vague ideals, and investigated their properties. They also provided conditions for a vague set to be a vague ideal. They discussed characterizations of a vague ideal. Ahn et al. [1] introduced the notion of vague quick ideals of BCK/BCI-algebras, and discussed related properties. Lee et al. [6] introduced the notions of vague d-subalgebras, vague d-ideals, vague $d^\sharp$-ideals and vague d^*-ideals. They established relations between vague d-subalgebras, vague BCK-ideals, vague d-ideals, vague $d^\sharp$-ideals and vague d^*-ideals. In this paper, we also use the notion of vague set in the sense of Gau and Buehrer to discuss the vague theory on BCK/BCI-algebras. We introduce the notion of positive implicative vague ideals in BCK-algebras, and then we investigate their properties. We investigate a relation between a vague ideal and a positive implicative vague ideal. We establish characterizations of a positive implicative vague ideal.

2. Preliminaries

We review some definitions and properties that will be useful in our results.

By a *BCI-algebra* we mean an algebra $(X, *, 0)$ of type (2,0) satisfying the following conditions:

(a1) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,
(a2) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,
(a3) $(\forall x \in X)\ (x * x = 0)$,
(a4) $(\forall x, y \in X)\ (x * y = 0,\ y * x = 0 \Rightarrow x = y)$.

A BCI-algebra X satisfying the additional condition:

(a5) $(\forall x \in X)\ (0 * x = 0)$

is called a *BCK-algebra*. In any BCK/BCI-algebra X one can define a partial order "$\leq$" by putting $x \leq y$ if and only if $x * y = 0$.

A BCK/BCI-algebra X has the following properties:

(b1) $(\forall x \in X)\ (x * 0 = x)$.
(b2) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$.
(b3) $(\forall x, y, z \in X)\ (x \leq y \Rightarrow x * z \leq y * z,\ z * y \leq z * x)$.
(b4) $(\forall x, y, z \in X)\ ((x * z) * (y * z) \leq x * y)$.

In particular, if X is a BCK-algebra then the following property hold:

(b5) $(\forall x, y \in X)\ ((x * y) * x = 0)$.

A subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ whenever $x, y \in S$. A subset A of a BCK-algebra X is called an *ideal* of X if it satisfies:

(c1) $0 \in A$,
(c2) $(\forall x \in A)\ (\forall y \in X)\ (y * x \in A \Rightarrow y \in A)$.

Note that every ideal A of a BCK/BCI-algebra X satisfies:

$$(\forall x \in A)\ (\forall y \in X)\ (y \leq x \Rightarrow y \in A). \tag{2.1}$$

A subset A of a BCK-algebra X is called a *positive implicative ideal* of X if it satisfies (c1) and

(c3) $(\forall x, y, z \in A)\ ((x * y) * z \in A,\ y * z \in A \Rightarrow x * z \in A)$.

Note that any positive implicative ideal is an ideal, but the converse is not true in general.

Lemma 2.1. [8] *Let X be a BCK-algebra. Then an ideal A of X is positive implicative if and only if it satisfies:*

$$(\forall x, y \in X)\ ((x * y) * y \in A \Rightarrow x * y \in A). \tag{2.2}$$

We refer the reader to the books [4] and [8] for further information regarding BCK/BCI-algebras.

Definition 2.2. [2] A *vague set* A in the universe of discourse U is characterized by two membership functions given by:

(1) A true membership function

$$t_A : U \to [0, 1],$$

and

(2) A false membership function

$$f_A : U \to [0, 1],$$

where $t_A(u)$ is a lower bound on the grade of membership of u derived from the "evidence for u", $f_A(u)$ is a lower bound on the negation of u derived from the "evidence against u", and

$$t_A(u) + f_A(u) \leq 1.$$

Thus the grade of membership of u in the vague set A is bounded by a subinterval $[t_A(u), 1 - f_A(u)]$ of $[0, 1]$. This indicates that if the actual grade of membership of u is $\mu(u)$, then

$$t_A(u) \leq \mu(u) \leq 1 - f_A(u).$$

The vague set A is written as

$$A = \{\langle u, [t_A(u), f_A(u)]\rangle \mid u \in U\},$$

where the interval $[t_A(u), 1 - f_A(u)]$ is called the *vague value* of u in A, denoted by $V_A(u)$.

Recall that if $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$ are two subintervals of $[0, 1]$, we can define a relation between I_1 and I_2 by $I_1 \succeq I_2$ if and only if $a_1 \geq a_2$ and $b_1 \geq b_2$. For $\alpha, \beta \in [0, 1]$ we now define (α, β)-cut and α-cut of a vague set.

Definition 2.3. [2] Let A be a vague set of a universe X with the true-membership function t_A and the false-membership function f_A. The (α, β)-*cut* of the vague set A is a crisp subset $A_{(\alpha,\beta)}$ of the set X given by

$$A_{(\alpha,\beta)} = \{x \in X \mid V_A(x) \succeq [\alpha, \beta]\}.$$

Clearly $A_{(0,0)} = X$. The (α, β)-cuts of the vague set A are also called *vague-cuts* of A.

Definition 2.4. [2] The α-*cut* of the vague set A is a crisp subset A_α of the set X given by $A_\alpha = A_{(\alpha,\alpha)}$.

Note that $A_0 = X$, and if $\alpha \geq \beta$ then $A_\alpha \subseteq A_\beta$ and $A_{(\alpha,\beta)} = A_\alpha$.

Equivalently, we can define the α-cut as

$$A_\alpha = \{x \in X \mid t_A(x) \geq \alpha\}.$$

3. Positive implicative vague ideals

For our discussion, we shall use the following notations on interval arithmetic:

Let $I[0, 1]$ denote the family of all closed subintervals of $[0, 1]$. We define the term "imax" to mean the maximum of two intervals as

$$\text{imax}(I_1, I_2) := [\max(a_1, a_2), \max(b_1, b_2)],$$

where $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2] \in I[0, 1]$. Similarly we define "imin". The concepts of "imax" and "imin" could be extended to define "isup" and "iinf" of infinite number of elements of $I[0, 1]$.

It is obvious that $L = \{I[0,1], \text{isup}, \text{iinf}, \succeq\}$ is a lattice with universal bounds $[0,0]$ and $[1,1]$ (see [1]).

In what follows let X denote a BCK-algebra unless otherwise specified.

Definition 3.1. [7] A vague set A of X is called a *vague BCK/BCI-algebra* of X if the following condition is true:

$$(\forall x, y \in X)\ (V_A(x * y) \succeq \text{imin}\{V_A(x), V_A(y)\}), \tag{3.1}$$

that is,

$$\begin{array}{l} t_A(x * y) \geq \min\{t_A(x), t_A(y)\}, \\ 1 - f_A(x * y) \geq \min\{1 - f_A(x), 1 - f_A(y)\} \end{array} \tag{3.2}$$

for all $x, y \in X$.

Definition 3.2. [7] A vague set A of X is called a *vague ideal* of X if the following conditions are true:

(d1) $(\forall x \in X)\ (V_A(0) \succeq V_A(x))$,
(d2) $(\forall x, y \in X)\ (V_A(x) \succeq \text{imin}\{V_A(x * y), V_A(y)\})$,

that is,

$$t_A(0) \geq t_A(x),\ 1 - f_A(0) \geq 1 - f_A(x), \tag{3.3}$$

and

$$\begin{array}{l} t_A(x) \geq \min\{t_A(x * y), t_A(y)\}, \\ 1 - f_A(x) \geq \min\{1 - f_A(x * y), 1 - f_A(y)\} \end{array} \tag{3.4}$$

for all $x, y \in X$.

Proposition 3.3. [7] *Every vague ideal A of X satisfies:*

$$(\forall x, y \in X)\ (x \leq y \Rightarrow V_A(x) \succeq V_A(y)). \tag{3.5}$$

Proposition 3.4. [7] *Every vague ideal A of X satisfies:*

$$(\forall x, y, z \in X)\ (V_A(x * z) \succeq \text{imin}\{V_A((x * y) * z), V_A(y)\}). \tag{3.6}$$

Proposition 3.5. *For a vague ideal A of X, the following conditions are equivalent:*

(1) $(\forall x, y \in X)\ (V_A(x * y) \succeq V_A((x * y) * y))$.
(2) $(\forall x, y, z \in X)\ (V_A((x * z) * (y * z)) \succeq V_A((x * y) * z))$.

Proof. Assume that A satisfies the condition (1). Since

$$((x * (y * z)) * z) * z = ((x * z) * (y * z)) * z \leq (x * y) * z$$

for all $x, y, z \in X$, we have $t_A((x * y) * z) \leq t_A(((x * (y * z)) * z) * z)$ and

$$1 - f_A((x * y) * z) \leq 1 - f_A(((x * (y * z)) * z) * z)$$

for all $x, y, z \in X$. It follows from (b2) and (1) that

$$\begin{aligned} t_A((x * z) * (y * z)) &= t_A((x * (y * z)) * z) \\ &\geq t_A(((x * (y * z)) * z) * z) \geq t_A((x * y) * z) \end{aligned}$$

$*$	0	a	b
0	0	0	0
a	a	0	0
b	b	b	0

Table 1. $*$-operation for X

and

$$\begin{aligned} 1 - f_A((x*z)*(y*z)) &= 1 - f_A((x*(y*z))*z) \\ &\geq 1 - f_A(((x*(y*z))*z)*z) \geq 1 - f_A((x*y)*z). \end{aligned}$$

Therefore A satisfies the second condition. Conversely, if we put $z = y$ in (2) and use (a3) and (b1), then we obtain the condition (1). This completes the proof. □

Definition 3.6. A vague set A of X is called a *positive implicative vague ideal* of X if it satisfies (d1) and

(d3) $(\forall x, y, z \in X)$ $(V_A(x*z) \succeq \text{imin}\{V_A((x*y)*z), V_A(y*z)\})$.

Note that (d3) is equivalent to the following assertion:

$$(3.7)\quad \begin{aligned} &t_A(x*z) \geq \min\{t_A((x*y)*z), t_A(y*z)\}, \\ &1 - f_A(x*z) \geq \min\{1 - f_A((x*y)*z), 1 - f_A(y*z)\} \end{aligned}$$

for all $x, y, z \in X$.

Example 3.7. Let $X = \{0, a, b\}$ be a BCK-algebra in which the $*$-operation is given by Table 1. Let A be a vague set in X defined by

$$A = \{\langle 0, [0.7, 0.2]\rangle, \langle a, [0.5, 0.3]\rangle, \langle b, [0.4, 0.4]\rangle\}.$$

It is routine to verify that A is a positive implicative vague ideal of X.

Theorem 3.8. *Every positive implicative vague ideal is a vague ideal.*

Proof. Let A be a positive implicative vague ideal of X. If we take $z = 0$ in (d3) and use (b1), then we obtain (d2). Hence A is a vague ideal of X. □

The following example shows that the converse of Theorem 3.8 may not be true.

Example 3.9. Consider a BCK-algebra $X = \{0, a, b, c\}$ with Cayley table which is given by Table 2. Let A be a vague set in X defined by

$$A = \{\langle 0, [0.8, 0.1]\rangle, \langle a, [0.7, 0.2]\rangle, \langle b, [0.7, 0.2]\rangle, \langle c, [0.5, 0.4]\rangle\}.$$

It is routine to verify that A is a vague ideal of X. But it is not a positive implicative vague ideal of X since

$$V_A(b*a) \not\succeq \text{imin}\{V_A((b*a)*a), V_A(a*a)\}.$$

It is natural to ask what is the condition under which a vague ideal is a positive implicative vague ideal? We now answer this question.

Theorem 3.10. *For a vague ideal A of X, the following are equivalent:*

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Table 2. Cayley table for X

(1) *A is a positive implicative vague ideal of X.*
(2) *A satisfies the condition* (1) *in Proposition 3.5.*

Proof. Assume that A is a positive implicative vague ideal of X. If we put $z = y$ in (d3) and use (a3) and (d1), then

$$\begin{aligned} V_A(x*y) &\succeq \text{imin}\{V_A((x*y)*y), V_A(y*y)\} \\ &= \text{imin}\{V_A((x*y)*y), V_A(0)\} \\ &= V_A((x*y)*y) \end{aligned}$$

for all $x, y \in X$.

Conversely, suppose that A satisfies the condition (1) of Proposition 3.5. Note that $((x*z)*z)*(y*z) \leq (x*y)*z$ for all $x, y, z \in X$. Using Proposition 3.5(1), (d2) and Proposition 3.3, we have

$$\begin{aligned} V_A(x*z) &\succeq V_A((x*z)*z) \\ &\succeq \text{imin}\{V_A(((x*z)*z)*(y*z)), V_A(y*z)\} \\ &\succeq \text{imin}\{V_A((x*y)*z), V_A(y*z)\}, \end{aligned}$$

and so A is a positive implicative vague ideal of X. □

Theorem 3.11. *For a vague ideal A of X, the following are equivalent:*

(1) *A is a positive implicative vague ideal of X.*
(2) *A satisfies the condition* (2) *in Proposition 3.5.*

Proof. Assume that A is a positive implicative vague ideal of X. Combining Theorems 3.8 and 3.10 and Proposition 3.5, A satisfies the condition (2) in Proposition 3.5.

Conversely, suppose that (2) is valid. For any $x, y, z \in X$, we have

$$\begin{aligned} V_A(x*z) &\succeq \text{imin}\{V_A((x*z)*(y*z)), V_A(y*z)\} \\ &\succeq \text{imin}\{V_A((x*y)*z), V_A(y*z)\}. \end{aligned}$$

Therefore A is a positive implicative vague ideal of X. □

Theorem 3.12. *For a vague set A in X, the following are equivalent:*

(1) *A is a positive implicative vague ideal of X.*
(2) *A satisfies conditions* (d1) *and*

$$(\forall x, y, z \in X)\ (V_A(x*y) \succeq \text{imin}\{V_A(((x*y)*y)*z), V_A(z)\}). \tag{3.8}$$

Proof. Assume that A is a positive implicative vague ideal of X. Then A is a vague ideal of X by Theorem 3.8, and so A satisfies the condition (d1). Using (d2), (b2), (a3), (b1) and Theorem 3.11, we have

$$\begin{aligned} V_A(x*y) &\succeq \text{imin}\left\{V_A\left((x*y)*z\right), V_A(z)\right\} \\ &= \text{imin}\left\{V_A\left(((x*z)*y)*(y*y)\right), V_A(z)\right\} \\ &\succeq \text{imin}\left\{V_A\left(((x*z)*y)*y\right), V_A(z)\right\} \\ &= \text{imin}\left\{V_A\left(((x*y)*y)*z\right), V_A(z)\right\}. \end{aligned}$$

Therefore (3.8) is valid.

Conversely, let A satisfy conditions (d1) and (3.8). For any $x, y \in X$, we have

$$\begin{aligned} V_A(x) = V_A(x*0) &\succeq \text{imin}\left\{V_A\left(((x*0)*0)*y\right), V_A(y)\right\} \\ &= \text{imin}\left\{V_A(x*y), V_A(y)\right\}. \end{aligned}$$

Hence A is a vague ideal of X. If we put $z = 0$ in (3.8), then

$$V_A(x*y) \succeq \text{imin}\left\{V_A\left(((x*y)*y)*0\right), V_A(0)\right\} = V_A\left((x*y)*y\right)$$

for all $x, y \in X$. It follows from Theorem 3.10 that A is a positive implicative vague ideal of X. □

Combining the above results, we have characterizations of a positive implicative vague ideal.

Theorem 3.13. *For a vague set A in X, the following are equivalent:*

(1) *A is a positive implicative vague ideal of X.*
(2) *A is a vague ideal of X satisfying the condition* (1) *in Proposition 3.5.*
(3) *A is a vague ideal of X satisfying the condition* (2) *in Proposition 3.5.*
(4) *A satisfies the conditions* (d1) *and* (3.8)

Lemma 3.14. [7] *For a vague set A in X, the following are equivalent:*

(1) *A is a vague ideal of X.*
(2) *A satisfies the following implication:*

$$(3.9) \qquad (\forall x, y, z \in X)\ ((x*y)*z = 0 \ \Rightarrow\ V_A(x) \succeq \text{imin}\left\{V_A(y), V_A(z)\right\}).$$

Theorem 3.15. *For a vague set A in X, the following are equivalent:*

(1) *A is a positive implicative vague ideal of X.*
(2) *A satisfies the following implication:*

$$(3.10) \qquad (((x*y)*y)*a)*b = 0 \ \Rightarrow\ V_A(x*y) \succeq \text{imin}\left\{V_A(a), V_A(b)\right\}.$$

for all $x, y, a, b \in X$.

Proof. Assume that A is a positive implicative vague ideal of X. Then A is a vague ideal of X by Theorem 3.8. Let $x, y, a, b \in X$ be such that $(((x*y)*y)*a)*b = 0$. It follows from Theorem 3.10 and Lemma 3.14 that

$$V_A(x*y) \succeq V_A\left((x*y)*y\right) \succeq \text{imin}\left\{V_A(a), V_A(b)\right\}.$$

Therefore A satisfies (3.10).

Conversely, suppose that A satisfies the condition (3.10) and let $x, a, b \in X$ be such that $(x * a) * b = 0$. Then $(((x * 0) * 0) * a) * b = 0$, and so

$$V_A(x) = V_A(x * 0) \succeq \operatorname{imin}\{V_A(a), V_A(b)\}.$$

Using Lemma 3.14, we know that A is a vague ideal of X. Note that

$$(((x * y) * y) * ((x * y) * y)) * 0 = 0$$

for all $x, y \in X$. It follows from (3.10) that

$$V_A(x * y) \succeq \operatorname{imin}\{V_A((x * y) * y), V_A(0)\} = V_A((x * y) * y)$$

and so A is a positive implicative vague ideal of X by Theorem 3.10. □

Theorem 3.16. *For a vague set A in X, the following are equivalent:*

(1) *A is a positive implicative vague ideal of X.*
(2) *A satisfies the following implication:*

$$(((x * y) * z) * a) * b = 0 \ \Rightarrow\ V_A((x * z) * (y * z)) \succeq \operatorname{imin}\{V_A(a), V_A(b)\} \tag{3.11}$$

for all $x, y, z, a, b \in X$.

Proof. Suppose that A is a positive implicative vague ideal of X. Then A is a vague ideal of X by Theorem 3.8. Let $x, y, z, a, b \in X$ be such that $(((x*y)*z)*a)*b = 0$. Then $V_A((x * z) * (y * z)) \succeq V_A((x * y) * z) \succeq \operatorname{imin}\{V_A(a), V_A(b)\}$ by Theorem 3.11 and Lemma 3.14. Thus (3.11) is valid.

Conversely, if we put $z = y$ in (3.11), then

$$V_A(x * y) = V_A((x * y) * (y * y)) \succeq \operatorname{imin}\{V_A(a), V_A(b)\}$$

whenever $(((x * y) * y) * a) * b = 0$ for all $x, y, a, b \in X$. It follows from Theorem 3.15 that A is a positive implicative vague ideal of X. □

By the mathematical induction, the above two theorems have more general forms.

Theorem 3.17. *For a vague set A in X, the following are equivalent:*

(1) *A is a positive implicative vague ideal of X.*
(2) *A satisfies the following inequality:*

$$V_A(x * y) \succeq \operatorname{imin}\{V_A(a_1), \cdots, V_A(a_n)\} \tag{3.12}$$

whenever $(\cdots((x * y) * y) * a_1) * \cdots) * a_n = 0$ for all $x, y, a_1, \cdots, a_n \in X$.

Theorem 3.18. *For a vague set A in X, the following are equivalent:*

(1) *A is a positive implicative vague ideal of X.*
(2) *A satisfies the following inequality:*

$$V_A((x * z) * (y * z)) \succeq \operatorname{imin}\{V_A(a_1), \cdots, V_A(a_n)\} \tag{3.13}$$

whenever $(\cdots((x * y) * z) * a_1) * \cdots) * a_n = 0$ for all $x, y, z, a_1, \cdots, a_n \in X$.

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References

[1] S. S. Ahn, Y. U. Cho and C. H. Park, Vague quick ideals of BCK/BCI-algebras, Honam Math. J. 30 (2008) 65–74.

[2] R. Biswas, Vague groups, Internat. J. Comput. Cognition 4 (2006) 20–23.

[3] W. L. Gau and D. J. Buehrer, Vague sets, IEEE Transactions on Systems, Man and Cybernetics 23 (1993) 610–614.

[4] Y. Huang, BCI-algebra, Science Press, Beijing, 2006.

[5] Y. B. Jun and C. H. Park, Vague ideals of subtraction algebras, Int. Math. Forum 2 (2007) 2919–2926.

[6] K. J. Lee, Y. H. Kim and Y. U. Cho, Vague set theory based on d-algebras, J. Appl. Math. Informatics 26 (2008) 1221–1232.

[7] K. J. Lee, K. S. So and K. S. Bang, Vague BCK/BCI-algebras, J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math. 15 (2008) 297–308.

[8] J. Meng and Y. B. Jun, BCK-algebras, Kyungmoon Sa Co. Seoul, 1994.

[9] C. H. Park, Vague deductive systems of subtraction algebras, J. Appl. Math. Comput. 26 (2008) 427–436.

[10] L. A. Zadeh, Fuzzy sets, Inform. Control 8 (1965) 338–353.

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